ELSEVIER
Journal of Pure and Applied Algebra 105 (1995) 299–305
JOURNAL OF PURE AND APPLIED ALGEBRA

An algebraic construction for integral Cech cohomology

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Communicated by C.J. Mulvey; received 20 August 1994

Abstract

We present a cohomology construction for a pair consisting of a commutative algebra and a space associated to the algebra; the space has as points some subcollection of the saturated multiplicative sets of the algebra, and topology a generalized version of the Zariski topology. We show that these cohomology modules coincide with the integral Cech cohomology modules of a compact Hausdorff space in a special case. The theory of compact ringed spaces is an essential tool in the approach taken. We indicate briefly how this approach simplifies the proof of a long established connection between a cohomology construction for a commutative algebra and the Cech cohomology of the maximal ideal space of a commutative Gelfand ring (of which the real Cech cohomology of a compact Hausdorff space is a special case).

1991 Math. Subj. Class.: 13D03, 16E40

1. Introduction

It has long been known that the real Cech cohomology of a compact Hausdorff space X may be obtained by application of either of two related cohomology constructions for a commutative algebra, to $C(X)$, the algebra of continuous real valued functions on X [4, 7]. Both of these constructions are based on the additive Amitsur complex. Both in essence involve associating to a commutative algebra the maximal ideal space of the algebra. Considering the special relation between the maximal ideal space of $C(X)$ and X, it is not surprising that neither construction translates directly to the case of integral Cech cohomology of a compact Hausdorff space.

In this paper we present an algebraic cohomology construction for a pair consisting of a commutative algebra, and a space associated to the algebra, which yields the integral Cech cohomology of a compact Hausdorff space in a special case. As suggested above, an essential feature of the approach lies in having some leeway in choosing an appropriate space to associate to an algebra. For this purpose we have defined what we have called SMS spaces of the algebra. These spaces are a generalization of the prime ideal space of an algebra.

SSDI 0022-4049(94)00149-9

The technique we use to establish the connection between our cohomology construction and Cech cohomology provides an application of Mulvey's results on compact ringed spaces [5, 6]. We assume that the reader has some familiarity with these results.

A key result in Kitchen's work concerns the Kitchen cohomology of commutative Gelfand algebras [4]. This may also be viewed as a special case of our cohomology construction, and the techniques we use considerably simplify the proof of this result. In the last section of this paper we discuss this briefly.

2. Preliminaries

All rings considered in this paper are associative, commutative rings with unity.

Let A be a commutative R-algebra. F^pA denotes the $(p+1)$-fold tensor product $A \otimes A \otimes \cdots \otimes A$ (all tensor products are taken over R). Bold case is used for elements of F^pA when $p > 0$.

For each $p \geq 0$, and for $0 \leq i \leq p+1$, $d_i^p : F^pA \to F^{p+1}A$ is defined on basis elements by $d_i^p(a_0 \otimes a_1 \otimes \cdots a_p) = a_0 \otimes a_1 \otimes \cdots \otimes 1 \otimes \cdots \otimes a_p$ with 1 in the $(i+1)$th position. (We write d_i when no ambiguity can arise.) d_i is an R-algebra homomorphism for each i. The additive Amitsur coboundary, $\delta^p : F^pA \to F^{p+1}A$ is defined by $\delta^p(\boldsymbol{a}) = \sum_{0 \leq i \leq p+1} d_i^p(\boldsymbol{a})$. Again we write δ when no ambiguity can arise. δ is an R-module homomorphism, but not an algebra homomorphism.

3. SMS spaces

Let A be a commutative ring. Let X be any collection of saturated multplicative sets (SMS's) of the ring. There is a natural way to topologize X: take as a basis for the topology the collection of all subsets of the form $U_b = \{Q \in X \mid b \in Q\}$, as b ranges through all elements of A. It is straightforward to check that these sets do form a basis for a topology on X; the proof hinges on the fact that the elements of X are SMS's. Identifying a prime ideal of a ring with its complement, it is clear that this construction is a generalization of the Zariski topology on the collection of prime ideals of a ring.

We shall refer to a collection of SMS's of a ring, which does not contain the ring itself, and which is endowed with the topology defined above as an SMS space of the ring. The concept of an SMS space was suggested by Charles Watts.

4. The cohomology theory associated to an R-algebra with an SMS space.

We use a standard procedure for constructing a ringed space from an ideal space of a ring (cf. for example [6]) to obtain a differential sheaf of rings (and an associated differential presheaf) on an SMS space of a ring. We start by constructing in each dimension $p \geq 0$ an ideal space consisting of ideals of F^pA, indexed by points in the SMS space, as follows.

Let X be an SMS space of an R-algebra A. Fix $p \geq 0$. Let $Q \in X$. Associate to Q the collection of all elements in F^pA of the form $a \otimes a \otimes \cdots \otimes a$, where $a \in Q$. Clearly this is a multiplicative set. Denote the saturation of this multiplicative set by ${}^pQ$. (Notice that any element of the form $a_0 \otimes a_1 \otimes \cdots \otimes a_p$ with $a_i \in Q$ for $0 \leq i \leq p$ is in ${}^pQ$, for it is a divisor of $(a_0a_1 \ldots a_p) \otimes (a_0 a_1 \ldots a_p) \otimes \cdots \otimes (a_0a_1 \ldots a_p)$ which is in ${}^pQ$.) Let K_Q^p denote the kernel of localization of F^pA at ${}^pQ$. Notice that if $x \in K_Q^p$, then there is an element of the form $b \otimes b \otimes \cdots \otimes b$, with $b \in Q$, and $x(b \otimes b \otimes \cdots \otimes b) = 0$. This follows from the fact that ${}^pQ$ is the saturation of the multiplicative set consisting of all elements of F^pA of this form.

$(K_Q^p)_{Q \in X}$ is an ideal space, indexed by points in X, and we may associate a sheaf to this ideal space by the standard construction. This yields the following proposition:

Proposition 1. *In each dimension $p \geq 0$, there is a sheaf of rings, $\mathfrak{F}_X^p$ say, on X, with stalk at $Q \in X$ given by F^pA/K_Q^p and there is a monomorphism*

$$\frac{F^pA}{\bigcap_{Q \in X} K_Q^p} \to \mathfrak{F}_X^p(X)$$

(where $\mathfrak{F}_X^p(X)$ denotes the ring of global sections).

In each dimension $p \geq 0$, $\mathfrak{F}_X^p$ is the sheaf canonically associated to the presheaf Ψ_X^p defined on a typical basic open set U_b by

$$\Psi_X^p(U_b) = \frac{F^pA}{\bigcap_{b \in Q} K_Q^p}.$$

Proof. By Theorem 1.1 in [6] it suffices to show that for each $a \in F^pA$, the set $V(a) = \{Q \in X \mid a \in K_Q^p\}$ is open in X. In fact let $Q \in V(a)$, so that $a \in K_Q^p$. Then there is some $b \in Q$ with $b \otimes b \otimes \cdots \otimes b . a = 0$. But then U_b is an open neighbourhood of Q contained in $V(a)$, so that $V(a)$ is open as required. $\square$

Denote the module $F^pA/\bigcap_{Q \in X} K_Q^p$ of global sections of Ψ_X^p by $C^p(A, R, X)$.

Proposition 2. *The additive Amitsur coboundary in F^*A induces a well-defined presheaf coboundary homomorphism on $(\Psi_X^p)_{p \geq 0}$ (in particular on $(C^p(A, R, X)_{p \geq 0})$. Hence too the Amitsur coboundary induces a well-defined sheaf coboundary on the graded sheaf $(\mathfrak{F}_X^p)_{p \geq 0}$.*

Proof. Fix $p \geq 0$, and consider a typical basic open set U_a of X. Let $x \in \bigcap_{a \in Q} K_Q^p$, so that for each Q with $a \in Q$ there is some $b_Q \in Q$ with $x(b_Q \otimes b_Q \otimes \cdots \otimes b_Q) = 0$. But then for each i (since d_i is an algebra homomorphism), $d_i(x)d_i(b_Q \otimes b_Q \otimes \cdots \otimes b_Q) = 0$. From the definition of d_i, it is clear that $d_i(b_Q \otimes b_Q \otimes \cdots \otimes b_Q) \in {}^{p+1}Q$, so that $d_i(x) \in K_Q^{p+1}$. Thus $d_i(x) \in \bigcap_{a \in Q} K_Q^{p+1}$. From $\delta x = \Sigma\, d_i(x)$, it follows then that $\delta x \in \bigcap_{a \in Q} K_Q^{p+1}$.

Thus the Amitsur coboundary induces a well-defined coboundary homomorphism on the graded presheaf $(\Psi^p)_{p \geq 0}$, as required. It follows from standard theory that the Amitsur coboundary induces a well-defined sheaf coboundary on $(\mathfrak{F}_X^p)_{p \geq 0}$. $\square$

This proposition indicates that we can construct a cohomology theory for any pair consisting of an R-algebra A, and a collection X of SMS's of the ring A, based on the cochain complex $C^p(A, R, X)$. Denote the q'th cohomology module of this complex by $H^q(A, R, X)$.

Clearly the monomorphisms of Proposition 1 yield a monomorphism $C^*(A, R, X) \to \mathfrak{F}_X^*(X)$ of cochain complexes. The following lemma, in conjuction with Mulvey's results on compact ringed spaces ([5] or [6]), provides a sufficient condition for these monomorphisms to be epimorphisms.

Lemma 3. *Let X be a compact, Hausdorff SMS space of a ring A. If $K_Q^0 + K_S^0 = A$ for each $Q, S \in X$ with $Q \neq S$, then for each $p \geq 0$, $C^p(A, R, X) \to \mathfrak{F}_X^p(X)$ is a compact representation of the ring $C^p(A, R, X)$.*

In fact it is easy to prove a stronger version of this lemma, since the condition $K_Q^0 + K_S^0 = A$ for each $Q, S \in X$ with $Q \neq S$ implies that X is Hausdorff; we do not prove this since in our main application (to Integral Cech cohomology) it will be clear for other reasons that X is Hausdorff.

Proof. Using Mulvey's equivalent condition for a representation to be compact [6, Theorem 3.3], it suffices to show that $K_Q^p + K_S^p = F^pA$. But if $a_Q \in K_Q^0$ and $a_S \in K_S^0$ are elements with $a_Q + a_S = 1$, it is clear that $a_Q \otimes 1 \otimes \cdots \otimes 1 + a_S \otimes 1 \cdots \otimes 1 = 1 \in F^pA$. $\square$

5. Application: The integral Cechcohomology of a compact Hausdorff space

We now show that by appropriate choice of the algebra and SMS space, the cohomology modules defined above may be identified with the integral Cech cohomology of a compact Hausdorff space.

Let $\mathfrak{X}$ be a topological space, and R an integral domain. Let A denote the R-algebra of all R-valued functions on $\mathfrak{X}$. For each $x \in \mathfrak{X}$ let $Q_x = \{f \in A \mid f$ is non-zero everywhere on some open neighbourhood of $x\}$. Since R is an integral domain, Q_x is an SMS of A. Let $X = \{Q_x \mid x \in \mathfrak{X}\}$ and regard X as an SMS space of A. Notice that the topology of $\mathfrak{X}$ has been subsumed in the choice of SMS space. Also notice that Q_x is not in general the complement of an ideal. Therefore, the space X cannot in general be described in terms of some subset of Spec A, so that the more general concept of an SMS space is essential to our approach.

Proposition 4. *If $\mathfrak{X}$ is a T_0 space and R an integral domain, then the SMS space X defined above is homeomorphic to $\mathfrak{X}$.*

Proof. Consider the function $\zeta : \mathfrak{X} \to X$ defined by $\zeta(x) = Q_x$ for $x \in \mathfrak{X}$. ζ is continuous. For if $x \in \zeta^{-1}(U_f)$ for some $f \in A$, then $f \in Q_x$, so that there is some open neighbourhood, W say, of x on which f is everywhere non-zero. But then W is an open neighbourhood of x which is contained in $\zeta^{-1}(U_f)$.

Next, ζ is clearly an onto mapping. Further since A consists of *all* functions from $\mathfrak{X}$ to R, it is straightforward to check that ζ is one-to-one iff $\mathfrak{X}$ is T_0. The same fact provides an easy demonstration that ζ is an open map: for an arbitrary open set V of $\mathfrak{X}$, consider the function g which is non-zero at $x \in \mathfrak{X}$ iff $x \in V$; then $\zeta(U) = U_g$. Thus ζ is a homeomorphism. $\square$

Now construct the differential sheaf $\mathfrak{F}_X^*$ on the SMS space X, and the cohomology modules $H^p(A, R, X)$.

Lemma 5. *Let $\mathfrak{X}$ be a Hausdorff space, and R an integral domain. With A and X as above, $C^p(A, R, X) \to \mathfrak{F}_X^p(X)$ is a compact representation for each $p \geq 0$.*

Proof. By Lemma 3, it suffices to show that if $Q_x \neq Q_y$, then $K_{Q_x} + K_{Q_y} = A$. As a first step, Proposition 4 shows that X is homeomorphic $\mathfrak{X}$, so that $Q_x \neq Q_y$ implies that $x \neq y$. Therefore, since X is Hausdorff, there exist disjoint open neighbourhoods of x and y, respectively. But then it is elementary to define functions f and g in A with $f \in K_{Q_x}$, $g \in K_{Q_y}$ and $f + g = 1$, whence the desired result. $\square$

Next we need to show that the sheaf $\mathfrak{F}_X^*$ is acyclic in dimensions greater than 0. First, however, notice that for each $Q \in X$, R may be regarded as a subring of A/K_Q (for clearly R may be identified with a subring of A, and since R is an integral domain, $R \cap K_Q = \{0\}$). Since we are tensoring over R, it is easy to see that this inclusion provides an augmentation $R \to F^*A/K_Q^*$ of the cochain complex of stalks at Q.

Lemma 6. *If $\mathfrak{X}$ is a compact Hausdorff space, and R an integral domain, then for each $Q \in X$ the augmented cochain complex $R \to F^*A/K_Q^*$ is acyclic.*

Proof. We prove this by constructing a contracting homotopy for the cochain complex. This homotopy is based on the usual one used to demonstrate the acyclicity of the cochain complex of germs of Alexander–Spanier cochains.

Suppose that Q is Q_x; i.e. $Q = \{f \mid f$ is non-zero on a neighborhood of $x\}$, where $x \in \mathfrak{X}$.

For each $p \geq 0$, define $D^p : F^pA \to F^{p-1}A$ on generators by $D^p(a_0 \otimes a_1 \otimes \cdots \otimes a_p) = a_0(x)\ (a_1 \otimes a_2 \otimes \cdots \otimes a_p)$. It is easy to see that D^p is a well-defined R-module homomorphism.

We claim that D^p induces a well-defined R-module homomorphism $F^pA/K_Q^p \to F^{p-1}A/K_Q^{p-1}$.

For suppose that $\boldsymbol{a} = \sum_i a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{ip} \in K_Q^p$. Then there is some $b \in Q$ with $(b \otimes b \otimes \cdots \otimes b)\boldsymbol{a} = 0$. It follows that $D^p(b \otimes b \otimes \cdots \otimes b\boldsymbol{a}) = 0$, i.e. $\sum b(x)\, a_{i0}(x)\, ba_{i1} \otimes \cdots \otimes ba_{ip} = 0$, equivalently $(b(x)b) \otimes b \otimes \cdots \otimes bD^p(\boldsymbol{a}) = 0$. But since R is an integral domain, and $b \in Q_x$, it follows that $b(x)b$ is non-zero on a neighbourhood of x, whence $b(x)b \in Q$. Thus $D^p(\boldsymbol{a}) \in K_Q^{p-1}$, as required. Denote this induced morphism of cochain complexes by D. Direct calculation shows that $\delta D + D\delta = id$. Thus D is a contracting homotopy for the cochain complex $R \to F^*A/K_Q^*$. □

The following lemma provides the means to draw the results together to achieve the identification of $H^p(A, R, X)$ with the integral Cech cohomology of $\mathfrak{X}$. This lemma may easily be proved by considering the spectral sequence associated to a double complex in the usual manner, and we therefore omit the proof.

Lemma 7. *Let X be a paracompact topological space, $\mathfrak{F}^*$ a non-negative cochain complex of sheaves of modules on X. Suppose that*

(1) $\check{H}^p(X, \mathfrak{F}^q) = 0$ for $p \geq 1$, and for all q.

(2) $H^p(\mathfrak{F}^*) = 0$ *for* $p \geq 1$.

Then $H^p(\mathfrak{F}^*(X)) \cong \check{H}^p(X, H^0(\mathfrak{F}^*))$ for all p. ($\check{H}^p(X, \mathfrak{F}^q)$ denotes the Cech cohomology of X with coefficients in $\mathfrak{F}^q$).

We are now in a position to prove the main theorem of this section:

Theorem 8. *Let $\mathfrak{X}$ be a compact Hausdorff space. Let R be an integral domain, and let A and X be as defined as at the beginning of this section. Then $H^p(A, R, X) \cong \check{H}^p(\mathfrak{X}, R)$ for each $p \geq 0$. In particular this is true when R is the ring of integers.*

Proof. Lemma 5 shows that $C^p(A, R, X) \to \mathfrak{F}_X^p(X)$ is a compact representation for each $p \geq 0$. It follows from Mulvey's results [5] that each of these homomorphisms is an epimorphism, and also that $\mathfrak{F}_X^p$ is a soft sheaf for each $p \geq 0$. Since $\mathfrak{F}_X^p$ is a sheaf of rings, the latter implies that $\mathfrak{F}_X^p$ is a fine sheaf for each $p \geq 0$. This ensures that $\check{H}^p(X, \mathfrak{F}_X^p) = 0$ for $p \geq 1$, and for all q. We have already established the acyclicity of $\mathfrak{F}_X^*$ in Lemma 6. Thus by Lemma 7, $H^p(\mathfrak{F}_X^*(X)) \cong \check{H}^p(X, H^0(\mathfrak{F}_X^*))$ for each $p \geq 0$. Adducing the homeomorphism of Proposition 4, the characterization of $H^0(\mathfrak{F}_X^*)$ of Lemma 6, and the fact that the representation maps are epimorphisms, establishes the theorem. □

We note in passing that this theorem of course also applies to calculating the real Cech cohomology of a compact Hausdorff space $\mathfrak{X}$. However, this case may be more elegantly described in terms of the ring of continuous real valued functions on $\mathfrak{X}$, as discussed in the next section.

6. Application to commutative Gelfand algebras

First notice that using Lemma 7, the proof technique of Theorem 8, and the stronger version of Lemma 3, we can easily establish the following:

Theorem 9. *Let A be an R-algebra, and X a compact SMS space of A, with the property that whenever $Q, S \in X$, with $Q \neq S$ then $K_Q + K_S = A$. Suppose as well that for each $Q \in X$ the cochain complex F^*A/K_Q^* is acyclic in dimensions greater than 0. Then $H^p(A, R, X) \cong \check{H}^p(X, H^0(\mathfrak{F}_X^*))$ for all p.* $\square$

This theorem provides an alternate, simpler proof for a generalized version of Theorem 63 in [4] (Theorem 10 below). First recall that in the case of a commutative ring A, the ring is said to be Gelfand if for each pair of distinct maximal ideals m_1 and m_2 of A, there exists $f \notin m_1$ and $g \notin m_2$ with $fg = 0$.

Now suppose that A is a commutative R-algebra, and a Gelfand ring. Consider the SMS space $\max A$, the maximal ideal space of A. It is well known that $\max A$ is compact, and by Theorem 3.7 in [1] whenever $Q, S \in \max A$, with $Q \neq S$ then $K_Q + K_S = A$. A proof that F^*A/K_Q^* is acyclic in dimensions greater than 0 when A is Gelfand may be found in [4] (Proposition 61). We have then the following theorem (in Kitchen's original version the base ring R was required to be a field).

Theorem 10. *If A is a commutative R-algebra and a Gelfand ring, then $H^p(A, R, \max A) \cong \check{H}^p(\max A, R)$.*

As a final remark, since $C(X)$ with X compact Hausdorff is a commutative Gelfand ring, Theorem 10 encompasses the special case of providing a strictly algebraic construction for the real Cech cohomology of a compact Hausdorff space.

Acknowledgements

This work was partially supported by a graduate fellowship at the University of Rochester. I thank my advisor Charles Watts for suggesting the problem and for helpful discussions.

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